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Application of basic optics principles for the determination of effective limits of numerical diffraction methods

INGENIERÍA FÍSICA

Principios ópticos básicos aplicados al cálculo de los límites de métodos de difracción numérica

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Resumen

Para determinar el rango de aplicación de los métodos de difracción numérica espectro angular y transformada Fresnel-Fraunhofer, se han utilizado las nociones ópticas básicas del principio de Babinet y el concepto del número de zonas de Fresnel. Usualmente dicho límite se evalúa considerando el correcto muestreo de la respuesta al impulso en el espacio libre para cada método en su evolución desde la abertura hasta el infinito. En este trabajo se hace uso combinado del principio de Babinet y el número de zonas de Fresnel para determinar la fase que debe exhibir un campo óptico propagado numéricamente una distancia dada; la desviación de la fase del campo óptico del valor pronosticado constituye la métrica de evaluación de la validez del método de propagación. Los resultados obtenidos permiten concluir que el límite usado con frecuencia para dividir el rango de aplicación para los métodos de espectro angular y la transformada de Fresnel-Fraunhofer debe de ser revisado. Se propone un nuevo límite que considera el número de pixeles utilizados para muestrear correctamente un salto de fase de

Palabras clave: *Holografía digital, propagación numérica, teoría de difracción.*

Abstract

The range of application of the methods of angular spectrum and Fresnel-Fraunhofer transform to compute numerical diffraction is evaluated via the basic optics concepts of Babinet´s principle and Frenel´s zones number. Conventionally, such limit is determined by assessing the correct sampling of the impulse response of the free space for each method as it evolves from the aperture to infinity. In this paper we make combined use of Babinet´s principle and Fresnel´s zones number to determine the phase that an optical wave field must exhibit after being propagated a given distance; the deviation of the phase of the optical field from the forecasted value is the metric utilized for testing the validity of the propagation method. The results show that the limit of application of the methods angular spectrum and Fresnel-Fraunhofer transform must be revisited. We propose a new limit that accounts for the number of pixels utilized for the correct sampling of a phase jump.

Keywords: *Diffraction theory, digital holography, numerical propagation.*

1. Introduction

Many applications of technology and science require the numerical propagation of the optics fields. For example the numerical reconstruction of the digitally recorded holograms (Kreis, 2004; Schnars & Jueptner, 2005), synthesis of computer generated holograms (Pan et al., 2009), numerical correction of optical aberrations in microscopy systems (Colomb et al., 2006), optical encryption of information (Matoba & Javidi, 1999), among many others, are some of the applications that require the numerical computation of the propagation of optics fields. This calculation indicates that is necessary to evaluate the wave equation, which in its scalar version is reduced to some way of the diffraction integral (Goodman, 2005). The numerical calculation of the diffraction integral can be performed using different numerical methods of propagation of optical fields; the methods most used are the angular spectrum (Mann et al., 2005) and the Fresnel-Fraunhofer transform (Picart & Juan-chang, 2012). No matter which propagation method used, the numerical computation enforces the sampling theorem for both the complex fields and propagators, this in order to guarantee the correct propagation of the optical field (Goodman, 2005; Li & Picart, 2012). The condition of correct sampling of the propagation kernel needed in the computation of the diffraction integral in any of its representations, has been the selected parameter to find the limit of validity of the different methods of numerical propagation (Mendlovic et al., 1997; Li & Picart, 2012; Shen & Wang, 2006; Restrepo & Garcia-Sucerquia, 2010). Many studies that evaluate the validity range for the different methods of numerical propagation are based on the elimination of phenomena as aliasing or redundancy in the diffraction patterns (Sypek et al., 2003). In this work, we present the evaluation of the limits of validity of the numerical propagation methods through the reproduction of the results predicted in numerical experiments of diffraction by the fundamentals concepts of optics Babinet's principle and the number of Fresnel zones.

In this paper, in the section 2 a review of Babinet's principle and the number of Fresnel zones is done, after the propagation methods angular spectrum and Fresnel-Fraunhofer transform are explained and the utilized numerical experiment is described. In the section 3, we show and discuss the results obtained; finally, in the section 5 the conclusion obtained product of research is presented.

2. Methodology

An explanation of the optics concepts used to carry out the study in this article is done. First, Babinet's principle and the number of Fresnel zones is exposed. After, a brief development of both theories of the angular spectrum and Fresnel-Fraunhofer transform, and an explanation of numerical experiment is performed.

2.1 Babinet's principle

Babinet's principle was formulated by the French physicist, mathematician and astronomer Jacques Babinet. The principle states: considering a diffracted field due to an opening and its complement, the addition of both diffracted fields equals the diffracted field through free space (Born & Wolf, 2005). In *Figure 1* an illustration of Babinet´s principle is presented. For propagation in free space has been considered a diffracted field produced by a circular opening, U_0 (x_1, y_1, z) in the lower left portion of that figure. In the upper-left side, it is shown the diffraction experiment to produce U_1 (x_1, y_1, z), namely the diffraction by the opening; the proper for its complement is presented in the upper-right to create U_2 (x_1, y_1, z). In the bottom of *Figure 1* the set of optical fields that illustrated Babinet´s principle, are shown. The right-most in the lower part of *Figure 1* is pixel-wise addition of U_1 (x_1, y_1, z) and U_2 (x_1, y_1, z). This result, U_1 (x_1, y_1, z) + U_2 (x_1, y_1, z), matches U_0 (x_1, y_1, z). In other words, for U_1 (x_1, y_1, z) and U_2 (x_1, y_1, z) the diffracted fields through an opening and its complement at a distance z, respectively, equals the diffracted field through free space:

$$U_0(x_1, y_1, z) = U_1(x_1, y_1, z) + U_2(x_1, y_1, z) \qquad (1)$$

If one considers a particular point at the propagated field in free space equals to zero, $U_0(x_1^{'}, y_1^{'}, z) = 0$, then, from Eq. (1) one obtains $U_1(x_1^{'}, y_1^{'}, z) = -U_2(x_1^{'}, y_1^{'}, z)$; this result indicates that the diffracted fields generated by the opening and its complement are equal in magnitude with a phase difference of π.

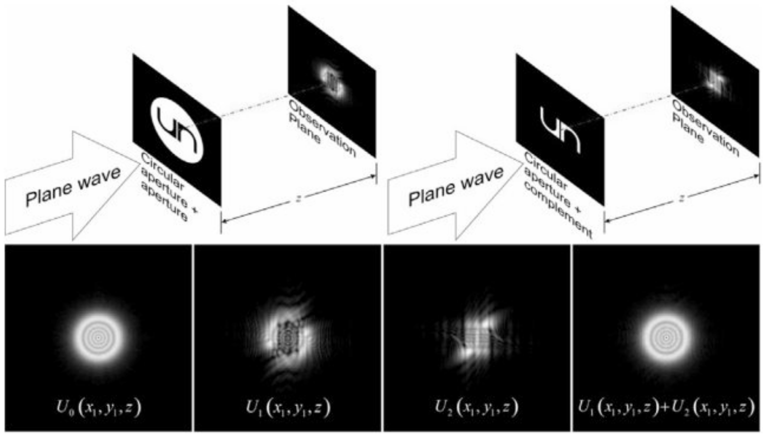

Figure 1. *Numerical simulation of Babinet's principle.*

2.2 Number of Fresnel's zones

The number of Fresnel zones is a postulated by the French physicist Augustin-Jean Fresnel. In its postulated, Fresnel proposes that the secondary spherical waves constituting a wavefront that propagates, as was stated by Huygens, can interfere each other (Born & Wolf, 2005). This interference means that the optical field is composed by sets of emitters that irradiate on average in the same phase for particular regions in space, denominated Fresnel's zones (Born & Wolf, 2005). The wavefields emitted by consecutive Fresnel's zones are radians out of phase, such that these consecutive zones interfere destructively. If it is considering a circular opening of radio illuminated by a spherical monochromatic wavefront of wavelength whit its point source at a distance from the center of the aperture and considering a plane of observation placed at a distance from the center of the aperture, the number of Fresnel's zones is given by:

$$N_f = \frac{R^2}{\lambda}\left(\frac{1}{a} + \frac{1}{b}\right) \tag{2}$$

Eq. (2) indicates from the conditions of the experimental setup can be predicted the value of the amplitude on the optical axis, just based on the information .provided by the number of Fresnel's zone; it can be predicted that when the system subtends an even number of Fresnel´s zones the optical axis will be dark and when this number is odd the same point will be bright Figure 2.

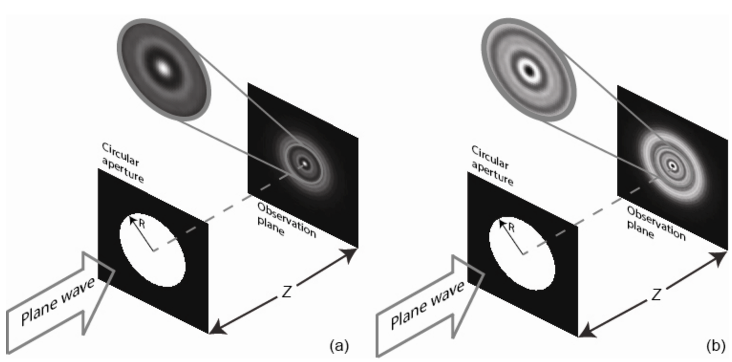

Figure 2. *Fresnel zones. a) Amplitude of optical fields for an odd number of Fresnel´szones. b)Amplitudeofopticalfieldsforanevennumber ofFresnel´szones*

2.3 Angular spectrum

To calculate the propagation of optics fields involves solving the wave equation:

$$\nabla^2 U_1(r,t) = \frac{1}{v^2}\frac{\partial^2 u_i(r,t)}{\partial t^2}; \quad \text{con i=x,y,z} \tag{3}$$

When the wavefront is expressed in the infinity base of plane waves propagating in all possible directions, the complex amplitude of the optical field propagated a distance z can be calculated through the expression:

$$U(x_1, y_1, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f_x, f_y, z)\exp\left[i2\pi\left(f_x x + f_y y\right)\right] df_x df_y \tag{4}$$

where, $A(f_x, f_y, z)$ is the angular spectrum in the plane z, that can be computed through $A(f_x, f_y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_1, y_1, z)\exp\left[-i2\pi\left(f_x x + f_y y\right)\right] dxdy.$ The kernel of pro- pa gation in Eq. (4) indicates an inverse Fourier transform, therefore Eq. (4), can be rewritten in terms of the Fourier transform operator $\Im^{-1}$ as:

$$U(x_1, y_1, z) = \Im^{-1}\left\{\Im\{U(x_0, y_0, 0)\}\exp\left[i\frac{2\pi}{\lambda}z\sqrt{1-\lambda^2\left(f_x^2+f_y^2\right)}\right]\right\}. \tag{5}$$

Eq. (5) allows the calculation of an optical field at a distance z from the aperture if one knows the optical field in the plane $(x_0, y_0, 0)$. However, the numerical computation of this equation requires of a discretization in terms of the input parameters; on considering that the input field can be discretized in a grid with pixels of sizes $\Delta x_0 \times \Delta y_0$, the continuous input coordinates are $x_0 = m\Delta x_0$ and $y_0 = n\Delta y_0$ with integer numbers. Furthermore, the output coordinates are discretized as $x_1 = p\Delta x_1$, $y_1 = q\Delta y_1$, $f_x = r\Delta f_x$ and $f_y = s\Delta f_y$ with *p, q, r* and *s* also integer numbers. Replacing the continue coordinates by the discrete coordinates, the integrals by summations and by using the discrete Fourier transform (DFT), Eq. (5) can be rewritten as:

$$U(p\Delta x_1, q\Delta y_1, z) = DFT^{-1}\left[DFT\left[U(m\Delta x_0, n\Delta x_0, 0)\right]\exp\left(i\frac{2\pi}{\lambda}z\sqrt{1-\lambda^2\left(\left(\frac{r}{M\Delta x_0}\right)^2+\left(\frac{s}{N\Delta y_0}\right)^2\right)}\right)\right] \tag{6}$$

2.4 Fresnel-Fraunhofer transform

Another way to solve the wave equation is through the Fresnel-Kirchhoff integral:

$$U(x_1, y_1, z) = -\frac{i}{2\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_0, y_0, 0)\frac{\exp(ikr)}{r}(1+\cos\chi)dx_0 dy_0 \tag{7}$$

where $r = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + z^2}$ is the distance from a point in the input plane $U = (x_0, y_0, 0)$ to a point in the output plane $U = (x_1, y_1, z)$. χ, is the angle between the vector r and the outward vector normal to the input plane. *Figure 3* illustrates this process of diffraction.

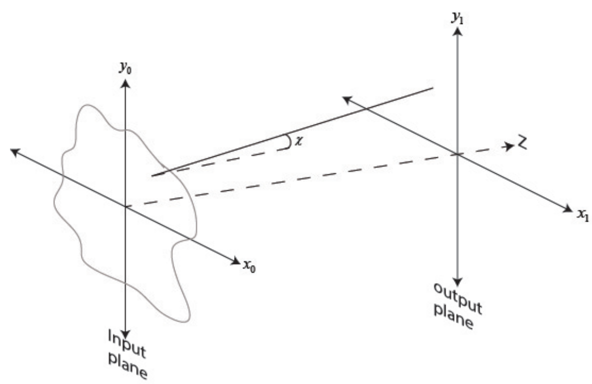

Figure 3. *Diagram of propagation between two planes.*

Using the paraxial approximation, by considering that the dimensions x_1, y_1 of output plane are much smaller than the distance z of propagation, r can be rewritten by means of a Taylor series expansion as $r = z + \frac{(x_1 - x_0)^2}{2z} + \frac{(y_1 - y_0)^2}{2z}$. Substituting this result in Eq. (7), one then gets:

$$U(x_1, y_1, z) = -\frac{i\exp(ikz)}{\lambda z}\exp\frac{ik}{2z}\left(x_1^2 + y_1^2\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_0, y_0, 0)$$
$$\exp\frac{ik}{2z}\left(x_0^2 + y_0^2\right)\exp\frac{-ik}{z}(x_0 x_1 + y_0 y_1)dx_0 dy_0 \tag{8}$$

Eq. (8) is known as the Fresnel-Fraunhofer approximation of the diffraction integral or also Fresnel-Fraunhofer transform (Goodman, 2005). To carry out the numerical calculation of Ec. (8), is necessary, as for the angular spectrum, to perform a process of discretization that allows expressing the Fresnel-Fraunhofer transform in a discrete fashion:

$$U(p\Delta x_1, q\Delta y_1, z) = \frac{-i\exp(ikz)}{\lambda z}\exp\frac{ikz}{2z}\left(\left(\frac{p\lambda z}{M\Delta x_0}\right)^2 + \left(\frac{q\lambda z}{N\Delta y_0}\right)^2\right)\times$$
$$DFT\left[U(m\Delta x_0, n\Delta y_0, 0)\exp\frac{ik}{2z}\left((m\Delta x_0)^2 + (n\Delta y_0)^2\right)\Delta x_0\Delta y_0\right] \quad (9)$$

Further details about the numerical diffraction processes can be read elsewhere Li & Picart (2012) or Kreis (2004).

2.5 Numerical experiment

From the analytical point of view the angular spectrum present no restriction in the distance of propagation, only the spatial frequencies must satisfy the exclusion of the evanescent waves, namely that $\lambda^2\left(f_x^2 + f_y^2\right) \leq 1$ (Goodman, 2005). The Fresnel-Fraunhofer transform presents a restriction inherited from the paraxial approximation performed on the Fresnel-Kirchhoff integral, where the condition $z^3 \gg \frac{1}{4\lambda}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]^2$ must be must be fulfilled. However, when the angular spectrum and Fresnel-Fraunhofer transform are numerically calculated, the sampling theorem (Goodman, 2005) must be met for the discretized equations of both propagation methods, Eq. (6) and Eq. (9). The correct sampling of the phases terms involved on the numerical computation of the wavefield propagation is controlled by the wavelength, the sizes of the input plane, and the propagation distance. Two of the more important studies in the literature about the range of application of each propagation method were presented by Mendlovic et al. (1997) and Sypek et al. (2003).

As can be observed in Eq. (6) and Eq. (9) the propagation distance affects the phase terms in the numerator for the angular spectrum and in the denominator for the Fresnel-Fraunhofer transform; this difference is a key factor to find the range of propagation for which are valid for the formalisms of angular spectrum and Fresnel-Fraunhofer transform. In the angular spectrum case two conditions must be fulfilled simultaneously: i) $1 \gg \lambda^2\left(\frac{r}{M\Delta x_0}\right)^2_{MAX}$ to guarantee that the computed waves propgate beyond the aperture, namely, they are not evanescent waves and ii) $\pi \geq \frac{2\pi}{\lambda}z\left[\sqrt{1-\lambda^2\left(\frac{r}{M\Delta x_0}\right)^2_{MAX}} - \sqrt{1-\lambda^2\left(\frac{r-P_N}{M\Delta x_0}\right)^2_{MAX}}\right]$ with P_N being the number of pixels needed to sample a π phase jump. Considering that the maximum integer index r along each rectangular direction is $M/2$, after a Taylor expansion up to second order of the latter equation it leads to a condition that must be satisfied to perform a correct sampling of the propagation kernel:

$$z \leq \frac{(M\Delta x_0)^2}{\lambda P_N (M - P_N)} \quad (10)$$

This condition imposes a limit up to where is correct to use the angular spectrum. The result shown in Eq. (10) is the same presented by Sypek et al. (2003) when the number of pixels $P_N = 1$, i.e. $z \leq \frac{M\Delta x_0^2}{\lambda}$ Similarly as we have shown for the case of angular spectrum, the critical factor for which one must guarantee the correct sampling, is the propagation phase factor within the kernel of Fresnel-Fraunhofer diffraction integral, namely the impulse response function for the free space. If one considers that at least P_N pixels are accounted for representing a π phase jump in the impulse function, one arrives to the limit condition of propagation

$$z \geq \frac{P_N \Delta x_0^2}{\lambda}[M - P_N] \quad (11)$$

this equations is equal to the expression reported by Sypek et al. (2003) when $P_N = 1$, i.e. $z \geq \frac{M\Delta x_0^2}{\lambda}$ The proposed numerical experiment consist in calculating the complex field of a plane wave when it is propagated a distance z after impinging on an opening and its complement to produce $U_1(x_1, y_1, z)$ and $U_2(x_1, y_1, z)$ respectively; the aperture and its complement are in contact with a circular opening of radio R as shown in Figure 1. In the observation plane is guaranteed an even number of Fresnel´s zones, in order to obtain a null propagated field $U_0(x_1, y_1, z) = 0$ on the optical axis. With this condition, we are able to guarantee that the amplitude of the optical field of the opening and its complement are equal in magnitude with a phase difference of π at the very optical field, as was predicted by Babinet's principle, at the very optical field, as was predicted by Babinet's principle, $U_1(x_1, y_1, z) = -U_2(x_1, y_1, z)$. This condition is used as metric for evaluating the range of validity of the propagation methods angular spectrum and Fresnel-Fraunhofer transform. The phase difference of the optical diffracted fields of the opening and its complement is measured while the propagation distance varies and the radio of the circular opening is modified to ensure that always an even number of Fresnel's zones. For both methods of numerical propagation angular spectrum and Fresnel-Fraunhofer transform, the propagation distance has been expressed in terms of the critical propagation distance Z_c Eq. (10) and Eq. (11), respectively. We additionally computed the derivative of the phase differences to increase the sensitivity in detecting any deviation of the predicted value.

3. Results and discussion

The results obtained for the angular spectrum are summarized in the Figure 4. In panel a) the plot with dot-lines corresponds to the phase difference calculated by subtracting the phases obtained for the opening and its complement on the optical axis for different distances of propagation; for this plot $P_N = 1.0$ in Eq. (10) leading to the expression proposed by Mendlovic $Z_c = M\Delta x_0^2 / \lambda$. The dash-line plot corresponds to the phases difference calculated when $P_N = 1.5$, which leads to $Z_c = 0.67(M\Delta x_0)^2 / \lambda(M-1.5)$. One can see in the panel a) that the phase difference when $P_N = 1.0$ is π from zero up to approximately a distance of propagation of 0.7 times z; beyond this point the phase difference on the optical axis presents oscillations that deviate the measured phase difference from that predicted by Babinet's principle. Panel b) shows the derivative from the phase difference of panel a). From a) and b) we can be concluded that for $P_N = 1.0$ appear perturbations on the phase of the wavefields propagated numerically that affect the transfer function implying a subsampling from kernel. However, when $P_N = 1.5$ the phase difference and its derivative remains constant for propagations from zero to z. Distances of propagation greater that z introduce perturbations in the phase difference measurement on the optical axis. These results indicate that to correctly sample the phase of the propagation kernel of the angular spectrum, at least three pixels must be used for each 2π phase jump, *Figure 5*. This indicate that to minimize any phase perturbation that affect the correct sampling in the propagation kernel of the angular spectrum the distance of propagation z should be:

$$z \leq \frac{0.67(M\Delta x_0)^2}{\lambda(M-1.5)}. \tag{12}$$

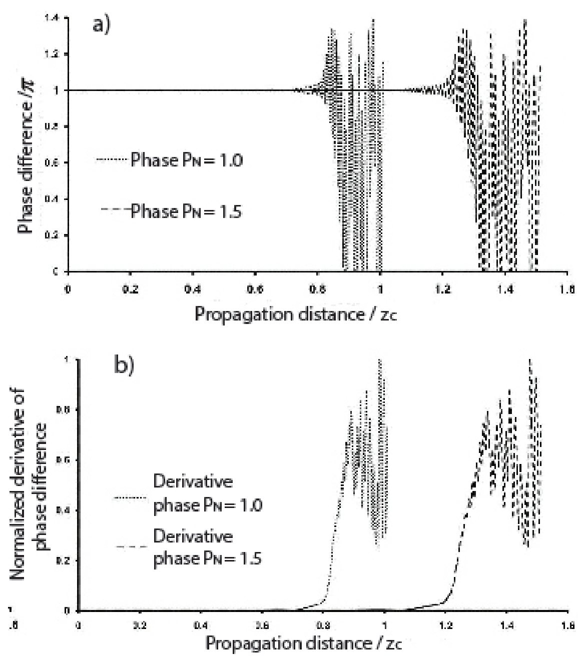

Figure 4. *Angular spectrum propagation method. a) Phase difference for the recorded wavefields of the aperture and its complement at different propagation distances for two different values of pixels P_N. b) Derivative of a) for both values of P_N. In these plots* $Z_c = (M\Delta x_0)^2 / \lambda P_N(M - P_N)$.

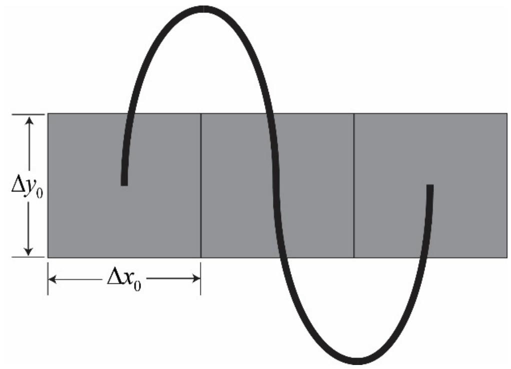

Figure 5. *Number of pixels needed to correctly sample of phase jump.*

Figure 6 shows the results obtained for the Fresnel-Fraunhofer transform by using the same methodology as in the angular spectrum. When $P_N = 1.0$, as it was expected for propagation distances shorter than the value of Z_c, the phase difference measurements on the optical axis are different to the value predicted by Babinet's principle until a distance of propagation 1.6 times Z_c. This observation is validated in panel b) where the derivative of the phase difference shown for $P_N = 1.0$. This behavior means that one single pixel is not enough to sample a π phase jump in the kernel of propagation. We have then increased the number of pixels to $N_p = 1.5$ allowing rewriting the value of the critical propagation as $Z_c = \left(1.5\Delta x_0^2 / \lambda\right)\left[M - 1.5\right]$. This result indicates that like in the angular spectrum, is necessary to make use of a minimum three pixels to correctly sample a 2π phase jump in the kernel of propagation of the Fresnel-Fraunhofer transform. Therefore, in order to minimize phase perturbations that affect the correct sample in the kernel of propagation of the Fresnel-Fraunhofer transform the distance of propagation should be:

$$Z \geq \frac{1.5\Delta x_0^2}{\lambda}\left[M - 1.5\right]. \qquad (13)$$

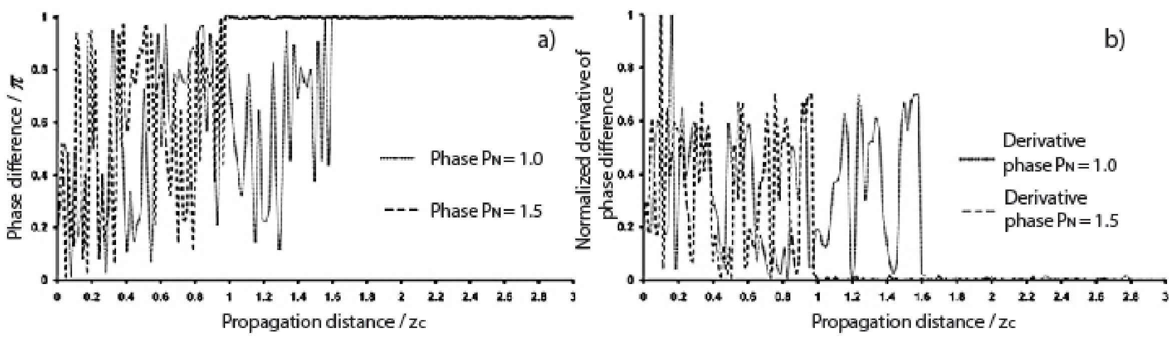

Figure 6. *Fresnel-Fraunhofer transform. a) Phase difference for the recorded wavefields of the aperture and its complement at different propagation distances for two different values of pixels PN. b) Derivative of a) for both values of PN. In these plots* $Z_c = \frac{P_N \Delta x_0^2}{\lambda}\left(M - P_N\right)$.

4. Conclusion

In this work, we have used the basic optics concepts Babinet's principle and the number of Fresnel's zones, in order to evaluate the limit up to where both numerical methods of optical field propagation, angular spectrum and Fresnel-Fraunhofer transform, can be used. As metric we have used the predicted phase by Babinet's theorem for a geometry that subtend an even number of Fresnel zones. The results obtained indicate that for the correct operation of the methods of angular spectrum and the Fresnel-Fraunhofer transform, is necessary to use at least three pixels to sample correctly phase jumps of in the kernel of propagation of each method. This result modifies the equation established in 1977 by Mendlovic et al. (1997) and ratified for Sypek et al. (2003).

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